Cosmology

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4 lectures in October 2003

1 History of the hot big bang model

Cosmology is an old subject. Some milestones:

200BC Eratosthenes gave first accurate measurement of the radius of the Earth, 10^7 metres, and hence distances to the moon and the sun, using lunar eclipses. The latter is 10^{11} metres.

1744 De Cheseaux - "Olber's paradox": why is the sky dark? Thus the Universe is not static, uniform and infinite.

1838 Bessel measured the distance to a star, 61 Cygni, using parallax, a distance of 10^{17} metres.

1915 Einstein's general theory of relativity, contained a repulsive cosmological constant term to maintain an apparently static Universe against gravitational collapse.

1920s Hubble measured the distance to Andromeda, 10^{22} metres, showing that it is a galaxy in its own right. He also found that distant galaxies are receding at a rate proportional to their distance. The size of the visible Universe was then measured to be roughly 10^{26} metres. Einstein retracted the cosmological constant.

1949 Alpher and Hermann predicted the cosmic background radiation; discovered in 1965 by Penzias and Wilson. This supported the big bang model of a Universe that was originally hot, and disproved the rival steady state theory.

1992 Small anisotropies in the CMB measured by the COBE satellite.

2001- At present, the WMAP probe is refining these measurements in sensivity and resolution. These measurements indicate that a cosmological constant (or similar effect) is actually present, causing an acceleration of the expansion.

The evidence for the standard hot big bang model comes not only from the observed expansion and the CMB, but also nucleosynthesis (nuclear reactions resulting in the present distribution of elements in material not processed by stars), and the connection between early fluctuations and the observed galaxy distribution. There are a number of questions and problems which we will come to later.

2 Units

In these lectures, we will set $G = 6.7 \times 10^{-11} Nm^2kg^{-2}$ and $c = 3.0 \times 10^8 ms^{-1}$ equal to unity. This is achieved, for example by converting times to lengths ct, and masses to lengths GM/c^2.

Problem

1 Show that G and c can be combined to construct a unit of power (energy per unit time), and give its value in SI units. What physical interpretation do you expect this to have?

3 Newtonian derivation of the Friedmann equation

The basic principles of cosmology follow from Newtonian gravity, without a detailed knowledge of relativity. This is because we can restrict attention to a small region near a "fixed" observer. Einstein's equivalence principle implies that the laws of physics in a "local inertial frame" of reference are given by those of non-GR physics, in this case due to Newton.

The cosmological principle states that all locations and directions are equivalent. This is, of course an approximation, in fact only good above about 10^{25} metres. Notice that the simplest picture, of galaxies receding at a rate proportional to their distance, satisfies this principle. That is, we have

$$\mathbf{x}(t) = \mathbf{x}_0 a(t)$$

where $a(t) = 0$ at the time of the big bang and $\mathbf{x}_0$ is constant for each galaxy. The scale factor $a(t)$ is linear for expansion at a constant rate. In a reference frame moving with a galaxy at $\mathbf{y}_0$ we measure a position $\mathbf{x}'$ given by

$$\mathbf{x}' = \mathbf{x} - \mathbf{y}_0 a(t)$$

and hence

$$\mathbf{x}'(t) = (\mathbf{x}_0 - \mathbf{y}_0) a(t)$$

which is just a translation of the galaxy positions.

The form of $a(t)$ must be determined by the gravitational attraction of the matter. The (spatially homogeneous) mass density is given by

$$\rho(t) = \rho_0 a(t)^{-3}$$

The acceleration of the matter at distance r from the origin is, assuming spherical symmetry and ignoring problems at infinity

$$\frac{d^2 r}{dt^2} = -\frac{4\pi\rho r^3/3}{r^2} = -\frac{4\pi\rho r}{3}$$

thus

$$a''(t) = -\frac{4\pi\rho_0}{3a(t)^2}$$

multiplying by $a'(t)$ and integrating, we have

$$a'(t)^2 - \frac{8\pi\rho_0}{3a(t)} = -K$$

Now, this is just an effective potential problem - if the constant K (negative of "total energy") is negative, the universe will continue to expand, but if it is positive, it will reach a maximum size and then contract.

The wavelength of any wave phenomena must increase proportional to $a(t)$ since otherwise some wave crests would be created or destroyed. This gives quantitatively the amount of redshift of light from distant galaxies.

$$z = \frac{\lambda_o}{\lambda_s} - 1 = \frac{a(t_o)}{a(t_s)} - 1 \approx \frac{a'(t_o)}{a(t_o)}(t_o - t_s) \approx \frac{a'(t_o)}{a(t_o)}\frac{d}{c}$$

where o indicates the observer and s the source. This proportionality between redshift and distance was what Hubble measured, and the constant (actually a function of time) a'/a is called Hubble's constant; the best value is now $71 \pm 4 kms^{-1}/Mpc = 2.3 \pm 0.1 \times 10^{-18} s$. Quantum mechanics tells us that a particle with momentum p is described by a wave with $\lambda = h/p$ where $h = 6.6 \times 10^{-34} Js$ is Planck's constant. Thus the momentum of a free particle (not just a photon) is proportional to $a(t)^{-1}$.

Now the energy density of light is proportional to $a(t)^{-4}$; this is because the photons are getting further apart (like the matter), but there is an extra factor of $a(t)^{-1}$ from the redshift. Recall that the energy of a photon is $E = hc/\lambda$. This would then lead to a cosomological equation of the form

$$a'(t)^2 - \frac{8\pi\rho_0}{3a(t)^2} = -K$$

Of course the Universe includes both matter and radiation; we can see that the radiation is more important at early times ("radiation dominated Universe") and matter is more important at later times ("matter dominated Universe"). At early times, we see that this equation predicts a big bang $a(t) = 0$ at some finite time in the past, which corresponds to infinite energy particles. The cosmic microwave background is the black body radiation from the era at which the Universe was hot enough to be opaque, redshifted to the present day. The temperature is about $2.73K$.

The third possible contribution to the energy density of the Universe is Einstein's cosmological constant Λ. We take this to have constant energy density, independent of time. Thus we can write

$$a'(t)^2 = \frac{8\pi}{3}\left[\frac{\rho_{0m}}{a(t)} + \frac{\rho_{0r}}{a(t)^2}\right] + \frac{\Lambda}{3}a(t)^2 - K$$

This is called the Friedmann equation. Note that a positive value of Λ can allow a static Universe (stationary point in the effective potential), but that this situation is unstable.

Note that it is possible to measure K if we know the other terms in the equation. $a'(t)^2/a(t)^2$ is the Hubble constant. The rest of the terms, say $8\pi\rho/3$ should be compared with this. Thus it is often written

$$\Omega = \frac{\rho}{\rho_c} = \frac{8\pi\rho a(t)^2}{3a'(t)^2}$$

The latest observations indicate that $\Omega \approx 1$ with ρ comprised of roughly 4% ordinary matter, 23% cold dark matter and 73% dark energy (eg cosmological constant).

Problems:

2 Solve the Friedmann equation, dropping insignificant terms, in the following regimes:

- **(a)** Radiation dominated: show that $a(t) = 0$ at some finite time in the past.
- **(b)** Matter dominated: find the present form of $a(t)$ assuming that Λ and K are small.
- **(c)** Final state: ignore everything except Λ. How might the Universe look at some time in the distant future?

3 If the Universe does recollapse (due to positive K), how do you expect the final state to differ from the initial state?

4 Relativity

If we want to measure distances in ordinary three dimensional space, we would write a "metric" like

$$ds^2 = dx^2 + dy^2 + dz^2$$

which means

$$s = \int ds = \int_a^b \sqrt{(\frac{dx}{d\lambda})^2 + (\frac{dy}{d\lambda})^2 + (\frac{dz}{d\lambda})^2} d\lambda$$

for a curve parametrised by λ. Linear transformations which preserve length are rotations. It is clear that the Newtonian laws of physics are invariant under a rotation; to put it another way, we can choose any system of (x, y, z) axes a long as they are orthogonal and right handed.

In special relativity there is a space-time metric

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2$$

Positive (in this convention) ds^2 ("timelike") measures the "proper" time on a clock moving through space-time. Thus an astronaut who moves close to the speed of light and then returns to Earth is younger than his twin who remained on Earth. Negative ds^2 ("spacelike") measures length. Zero ("null") means a photon or other object moving at the speed of light. Linear transformations which preserve the relativistic length are called Lorentz transformations. They consist of combinations of rotations and "boosts". The latter correspond to observers moving with different velocities. The speed of light is, however preserved by these transformations.

General relativity is a relativistic theory of gravity. In Newton's gravity, all objects fall with the same acceleration (ignoring air resistance etc), so that gravity can be viewed as an intrinsic property of the space, rather than of the falling object. In general relativity, space-time is curved, so distances are more complicated than given by the special relativistic formula above. Objects under gravity move so as to extremise (usually maximise) their proper time; such a path is called a "geodesic".

For example, the space-time metric corresponding to a weak, slow moving gravitational field can be written as

$$ds^2 = (1 + 2\Phi)dt^2 - (1 - 2\Phi)(dx^2 + dy^2 + dz^2)$$

where $\Phi(t, x, y, z)$ is the gravitational potential, for example $-M/r$ near a spherical mass. "Weak" here means $|\Phi| \ll 1$. This equation shows that clocks run more slowly near a massive object, and that there is an increase in the amount of space. Now, consider the path of a projectile, fixed to the surface of the Earth at times $t = 0$ and $t = T$. For intermediate times it will maximise its proper time by rising out of the gravitational potential. It cannot do this arbitrarily rapidly, however, because it will then suffer the special relativistic time dilation effect. The actual parabolic path of the projectile in space-time is the result of balancing these effects. Specifically, assuming $\Phi \ll 1$ and $dz/dt << 1$ we have proper time

$$d\tau/dt = 1 + \Phi - (dz/dt)^2/2 = L$$

where the L signifies the Lagrangian function since we want to find the stationary point of its time integral. This is achieved by the Euler Lagrange equations

$$\frac{\partial L}{\partial z} = \frac{d}{dt}\frac{\partial L}{\partial \dot{z}}$$

$$\frac{\partial \Phi}{\partial z} = \frac{d}{dt}(-\frac{dz}{dt})$$

which is just the Newtonian result.

Note that the length element may look quite different in a new coordinate system, even for the same space or space-time. For example, we can write

$$ds^2 = dx^2 + dy^2 + dz^2$$

in Cartesian coordinates, or

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin\theta d\phi^2$$

in spherical polar coordinates, defined by $x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi, z = r\cos\phi$. Note that (t, x, y, z) do not measure physical times lengths in general relativity; they are just coordinate labels. This raises the question - what is actually measurable in general relativity? Or, how does a curved spacetime physically differ from the flat spacetime of special relativity?

The answer, curvature, expresses itself in various ways:

1. Violations in the laws of Euclidean geometry, for example the sum of the angles of a triangle not adding to π.

2. Path dependent parallel transport of vectors, and hence no general method of comparing vectors at two different points.

3. Geodesics which are initially parallel do not remain so; on a sphere (positive curvature) they converge, while on a hyperbolic space (negative curvature) they diverge.

4. Coordinate invariant quantities calculated from the second derivatives of the metric.

The space-time curvature in general relativity is determined by the relativistic analogue of the mass density called the stress-energy tensor, containing energy density, energy flux and pressure.

$$G_{\mu\nu} - \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}$$

where $G^{\mu\nu}$ is the Einstein (curvature) tensor, given by a complicated formula involving the second derivatives of the metric, $g^{\mu\nu}$ is the metric and $T^{\mu\nu}$ is the stress-energy tensor. These equations are unique if we assume that the curvature is linear in the second derivative of the metric, and automatically satisfies energy- momentum conservation.

Problems

4 Find a Lorentz transformation that mixes the t and x components.

5 Find an example in which the path taken by a particle in a gravitational field is a non-maximal stationary point of the proper time.

5 Relativistic cosmology

Einstein's equations are very hard to solve in general, however if we assume homogeneity and isotropy, the problem simplifies greatly. It means that the set of points a given proper time t from the initial singularity form a surface of constant curvature, a sphere for positive curvature, Euclidean flat space for zero curvature, or a hyperbolic space for negative curvature. Such a space has metric

$$d\sigma^2 = \frac{dx^2 + dy^2 + dz^2}{\left[1 + \frac{K}{4}(x^2 + y^2 + z^2)\right]^2}$$

where K is the curvature. In polar coordinates it reads

$$d\sigma^2 = \frac{dr^2}{1 - Kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

or a transformation of the radial coordinate gives

$$d\sigma^2 = \left\{ \begin{array}{c} K^{-1} \\ 1 \\ -K^{-1} \end{array} \right\} \left[d\chi^2 + \left\{ \begin{array}{c} \sin^2\chi \\ \chi^2 \\ \sinh^2\chi \end{array} \right\} (d\theta^2 + \sin^2\theta d\phi^2) \right]$$

for $K > 0$, $K = 0$, $K < 0$ respectively. The Robertson-Walker metric, which takes account of the expansion, is then

$$ds^2 = dt^2 - a(t)^2 d\sigma^2$$

or a transformation of the time coordinate gives

$$ds^2 = a(\eta)^2 \left[d\eta^2 - d\sigma^2\right]$$

We compute the Einstein tensor corresponding to this metric, and find

$$G_{tt} = 3\frac{a'(t)}{a(t)} + 3\frac{K}{a(t)^2}$$

This is equal to $8\pi T_{tt} = 8\pi\rho$ where ρ is energy density as calculated before. Thus we reproduce the Friedmann equation. The rest of Einstein's equations only give a single equation by isotropy, relating derivatives of the expansion to the pressure. This is equivalent to the one we have by conservation of energy, which is automatically satisfied by Einstein's equations. Thus, without a cosmological constant, a positive curvature Universe will recollapse, while zero or negative curvature models will expand forever.

Note the distinction between curved space and curved space-time. A flat space-time has zero density, but in cosmological terms, the spaces of equal time from the origin are negatively curved.

Problem

6 Find the coordinate transformations defining χ and η.

6 Inflation

Difficulties/questions with the standard hot big bang model:

- The flatness problem - why is K so close to zero?
- The horizon problem - why is the Universe so uniform in causally disconnected regions?
- The cosmological constant problem - why is Λ so small (120 orders of magnitude less than naive predictions of quantum gravity)?
- What happens at the singularity?
- What is the global topology and why?
- Why the small but nonzero level of density fluctuations (leading to galaxies), but no exotic relics?

Derivation of the horizon: We solve the Friedmann equation with radiation or matter (assuming a sudden switch), and find

$$a(t) = \begin{cases} t^{1/2} & t < 1 \\ t^{2/3} & t > 1 \end{cases}$$

where various constants have been absorbed into a and t. The time $t = 1$ corresponds to the crossover between radiation and matter, and $a = 1$ corresponds to the scale factor at that time. Our own time is roughly $t = 150000$, $a = 3000$. Now, in terms of the conformal time variable η, the spacetime is simply

$$ds^2 = a^2(d\eta^2 - d\sigma^2)$$

so a photon travels a distance given by $\eta = \int dt/a$. Thus we compute

$$\eta = \int \frac{dt}{a} = \begin{cases} 2t^{1/2} & t < 1 \\ 3t^{1/3} - 1 & t > 1 \end{cases}$$

Now, the distance travelled by a photon (assuming the Universe was transparent) up to recombination is $\eta(4) = 3.8$ and the distance following recombination is $\eta(150000) - \eta(4) = 155$. Thus the horizon distance is a few percent of the maximal distance at opposite points in the sky.

There have been a number of attempts to treat the flatness, horizon and fluctuation problems using *inflation* in which additional scalar field(s), arising from quantum field theories, lead to a temporary exponential growth of the scale factor $a(t)$. In these models, the scenario is something like

1. Begin with scalar field ϕ having potential energy $V(\phi)$ relative to its minimum (equilibrium) value.

2. It causes an exponential increase in the scale factor through the Friedmann equation.

$$\frac{a'(t)^2}{a(t)^2} = \frac{8\pi}{3}\left[V(\phi) + \phi'(t)^2\right]$$

and its own equation of motion

$$\phi''(t) + 3\frac{a'(t)}{a(t)}\phi'(t) = -V'(\phi)$$

This evolution is supposed to be "slow" (ignore $\phi'(t)^2$ and $\phi''(t)$ terms) to allow the Universe to expand by some large exponential amount.

3. When it returns to its equilibrium value $V(\phi) = 0$ the "kinetic" energy $\phi'(t)^2$ is converted into ordinary particles in a "reheating" phase.

4. The standard hot big bang model proceeds as before.

The large amount of expansion solves the horizon problem, and also leads to predictions of the flatness and fluctuations in agreement with observations. The fluctuations in particular are predicted to be Gaussian in the cosmic microwave background observations. However it still does not answer the question of initial conditions - it just pushes it back to a quantum gravity regime that is not understood.

Problem

7 How does inflation help? Assuming an exponential expansion before the decoupling of matter and radiation, show that it is possible to solve the horizon problem.

7 Determination of cosmological parameters

This is obtained from the following sources of data:

1. The cosmic microwave background radiation. The data consists of measurements of the temperature and polarisation as a function of direction on the sky. The current temperature of the CMB is 2.73K, redshifted from the recombination temperature of about 3000K (ie $z \sim 1000$). Corrections are usually measured in terms of multipole moments, that is,

$$\frac{\delta T}{T} = \sum_l \sum_{m=-l}^{l} a_{lm} Y_l^m(\theta, \phi)$$

averaged over orientations,

$$C_l = \frac{1}{2l+1}\sum_{m=-l}^{l} |a_{lm}|^2$$

and then converted to a power density,

$$l(l+1)C_l/2\pi$$

The first $l = 1$ is the dipole, which is the Doppler effect due to the fact that we are moving relative to the “fixed galaxies” of the Friedmann model. It is $6.7mK$, indicating a velocity

$$v = \frac{6.7 \times 10^{-3}}{2.73}c = 700kms^{-1}$$

This dipole is used to calibrate the detector, and then subtracted to give fluctuations of cosmological origin. These are

- Sachs-Wolfe corrections due to gravitational time dilation and the time dependence of the gravitational potential.
- Temperature fluctuations in the recombination plasma
- Doppler shift - velocity fluctuations
- Later scattering events, “reionization” due to the first star formation.

The first three are strongly correlated: an increase in density will lead to fluctuations in gravitational potential, temperature and velocity. Light is polarized through scattering events, so polarization measurements give additional information about the latter.

At the largest scales (small l) the Sachs-Wolfe effect dominates, leading to a plateau in the spectrum, and giving information about the density fluctuations at different scales. At smaller scales peaks and troughs are observed, due to acoustic modes of oscillation, giving information about the content of the plasma (eg proportion of baryons). In addition, fairly early reionisation has been observed, indicating cold (rather than hot) dark matter is present to help the gas to clump, forming stars at $z \sim 20$. There is also an indication that the lowest $l = 2, 3$ terms are smaller than suggested by present theories, due to a nontrivial topology of the Universe. A recent paper in Nature suggested a dodecahedral topology in a positive curvature Universe, but the data is not conclusive.

2. Large scale structure observations (including the distribution of galaxies, velocities of galaxies, gravitational lensing using dark matter, reinforce information about the strength of fluctuations, but need numerical simulations for comparison since the formation of clusters of galaxies etc. is nonlinear.

3. Deuterium is produced in the big bang, but destroyed in stars. Thus observations of its abundance can put limits on parameters such as the baryon density.

4. The Hubble constant (present expansion rate) can be measured directly using distances and velocities of galaxies. It depends on the assumptions made in measuring distance.

5. Limits on the age of the Universe can be obtained from dating the oldest stars. Many of these are about 12Gyr, close to predictions from other sources.

All data need to be combined together to make a best fit to parameters. As noted above, this seems to be about 4% baryonic matter, 23% cold dark matter, 73% cosmological constant, zero curvature, Hubble constant 72 km/s/Mpc, age about 13.7Gyr.

Problems

8 Assuming that the Universe is now 27% matter and 73% cosmological constant, calculate the percentages at times when the Universe is (i) half, (ii) double, its current size.

9 Identify as many as possible assumptions made in estimating the cosmological parameters. In what ways could new data or theories affect the conclusions about current values of parameters and the cosmological evolution of the Universe?

8 Cosmological chaos

Chaos can occur in two places in general relativity, in particle trajectories and in the Einstein field equations. In both cases there is a fundamental issue to do with the coordinate freedom: Chaos must be defined in terms of coordinate invariant properties. For example the maximum Lyapunov exponent

$$\lambda_{\max} = \lim_{t\to\infty} \lim_{\epsilon\to 0} \frac{1}{t} \ln \frac{|f^t(x_0+\epsilon) - f^t(x_0)|}{\epsilon}$$

is defined in terms of the non-invariant quantity t; a different definition would give a different result, including 0 or ∞. In fact, this caused quite a lot of debate in the 80s and 90s for the Mixmaster cosmological model (below). It was resolved using fractal dimensions, which are coordinate invariant (technically diffeomorphism invariant).

Examples of chaos in the field equations are the Mixmaster cosmological model, which is the most general assuming homogeneity but allowing three different scale factors in the x, y, z directions. This leads to six coupled ODEs replacing the Friedmann equation (the minimum required for chaos is three). It is not realistic cosmologically, but is thought to describe general singularities. Chaos has also been observed in models containing a scalar field, similar to the inflation scenario considered previously.

Let us now turn to the case of photons propagating through the Friedmann universe. If the Universe is compact, its size in comoving coordinates is constant, so chaos in these coordinates has an invariant meaning. We will use Lyapunov exponents, noting that the stability and topology of periodic orbits are invariant. It is then reasonable to extend the definition to open universes. As in computing the horizon length, we remove the scale factor by going to conformal time η. And as in the Newtonian case, we introduce a gravitational potential using weak field and slow moving assumptions:

$$ds^2 = a^2(\eta)\{(1+2\Phi)d\eta^2 - \frac{1-2\Phi}{[1+\frac{K}{4}(x^2+y^2+z^2)]^2}(dx^2+dy^2+dz^2)\}$$

This is accurate to about ten percent - there are relativistic neutrinos and to a lesser extent gravitational waves.

The separation between two trajectories in relativity is given by the geodesic deviation equation. Using the Lagrangian formalism we see that the geodesics are described by an equation relating the second derivative of position with the first derivative of the metric coefficients. Looking at two closely spaced trajectories will involve another spatial derivative, leading to a term involving curvature, or in Newtonian terms, tidal forces. This is indeed the case: the geodesic deviation equation is

$$\frac{D^2\xi^\alpha}{d\lambda^2} = R^\alpha_{\mu\nu\beta}V^\mu V^\nu \xi^\beta$$

where ξ is the separation, R is the Riemann curvature tensor, and V is the tangent vector to the trajectories. Computing the curvature (a complicated task), comparing photons at the same η, and ignoring perturbations in the direction of motion which don't contribute, we find

$$\frac{d^2}{d\eta^2}\begin{pmatrix}\xi\\ \eta\end{pmatrix} = -\begin{pmatrix} K+2\Phi_{\xi\xi}+\Phi_{\zeta\zeta} & 2\Phi_{\xi\eta} \\ 2\Phi_{\xi\eta} & K+2\Phi_{\eta\eta}+\Phi_{\zeta\zeta}\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix}$$

Note that in the absence of perturbations, the equation just gives a sinusoidal function for positive curvature (the sphere), a straight line for zero curvature (flat) and a hyperbolic function for negative curvature. The Lyapunov exponent indicates chaos in the latter case. Note that one method of studying Hamiltonian systems is to convert the problem to geodesic motion on a curved space, and show the curvature is always negative.

The behaviour of the perturbed case is very interesting. Let us consider the one dimensional analogue (two dimensions doesn't change much). The equation

$$\frac{d^2u}{d\eta^2} = -(Af(\eta)-1)u. \tag{1}$$

describes an inverted pendulum with pivot moved according to the function $f(\eta)$. If $f(\eta)$ is a cosine, we have the Mathieu equation which has a very complicated behaviour called parametric resonance. A naturally stable system can become unstable if the frequency of the perturbation is related to the natural frequency of oscillation. Likewise a naturally unstable situation can be stabilised by a sufficiently rapid oscillating perturbation as first noted by Kapitsa. In the cosmological case the perturbation must be sufficient to change the sign of the curvature.

Let us determine conditions under which a stochastic perturbation can stabilise the motion. Split $u(\eta)$ into fast and slow components

$$u(\eta) = \langle u(\eta) \rangle + u_f(\eta)$$

where the average is over a time much greater than a typical frequency of the perturbation. Substitute into the equation of motion and average:

$$\frac{d^2 \langle u \rangle}{d\eta^2} = \langle u \rangle - A \langle f(\tau) u_f(\tau) \rangle$$

then subtract, noting that in the high frequency limit we have $Af \gg 1 \approx \langle u \rangle \gg u_f$.

$$\frac{d^2 u_f}{d\eta^2} \approx -Af(\tau) \langle u(\tau) \rangle$$

Now $\langle u \rangle$ is roughly constant, so we can integrate twice to get

$$\int f(\eta) = v(\eta)$$

$$\int v(\eta) = x(\eta)$$

where constants must be chosen to keep the averages of these quantities zero. Then we have

$$u_f = -Ax(\eta) \langle u \rangle$$

substituting into the previous expression we have

$$\frac{d^2 \langle u \rangle}{d\eta^2} = \langle u \rangle (1 - A^2 \langle f(\eta) x(\eta) \rangle) = \langle u \rangle (1 - A^2 \langle v(\eta)^2 \rangle)$$

Thus the stability criterion is that $A^2/(\omega^2 |K|)$ (now putting back the curvature) should be roughly greater than one. Note that the stochastic perturbation must not have low frequency components or this analysis may not hold. In the cosmological case, the density is homogeneous at large scales, so this is quite reasonable. The frequency and amplitude of fluctuations are also sufficient to see this stabilisation. These results show that, whatever the value of K (which is still somewhat uncertain), the photon trajectories are strongly stabilised. Stochastic stabilisation has many non-cosmological applications, which we are currently investigating. More details of this work are in the paper at

http://www.arxiv.org/abs/nlin.CD/0305056

Problem

10 Integrate the one dimensional equation numerically using noise given by a sum of many random cosines with a minimum frequency high compared to the constant expansion rate. Find how large the minimum frequency must be for your choice of random distribution, and check the above stabilisation criterion. A solution is given in the above paper.